Exponential and Logarithmic Functions

Use laws of exponents to derive laws of logarithms. Use laws of logarithms to solve problems. Solve exponential and logarithmic equations. Solve problems that can be modeled using equations involving exponents and logarithms.

Exploring Exponential Models

Analyze, describe and sketch graphs of exponential functions by examining intercepts, zeros, domain and range, and asymptotic and end behavior.

Exploring Exponential Models

• An **exponential function** is a function with the general form $y = ab^x$, where $x$ is a real number, $a \neq 0$, $b > 0$, $b \neq 1$.

• You can use an exponential function to model growth when $b > 1$. When $b > 1$, $b$ is the **growth factor**.

• An exponential function can be used to model decay when $0 < b < 1$. When $b < 1$, $b$ is a **decay factor**.

• An **asymptote** is a line that a graph approaches, but never reaches, as it moves away from the origin.

Example

• Graphing Exponential Growth
  – Graph $y = 2^x$.

Cont...

• Graph each function.
  – $y = 4(2)^x$
  – $y = 3^x$
Example

- Population – Refer to the graph. In 2000, the annual rate of increase in the U.S. population was about 1.24%.
  - Find the growth factor for the U.S. population.
  - Suppose the rate of increase continues to be 1.24%. Write a function to model U.S. population growth.

Cont...

- Predict U.S. population in 2015 to the nearest million.
- Explain why the model and your prediction may not be valid for 2015.
- Suppose the rate of population increase changes to 1.4%. Write a function to model population growth and use it to predict the 2015 population to the nearest million.

Example

- Writing an Exponential Function
  - Write an exponential function \( y = ab^x \) for a graph that includes (2,2) and (3,4).

Example

- Analyzing a Function
  - Without graphing, determine whether the function \( y = 14(0.95)^x \) represents exponential growth or exponential decay.

Investigation: Tournament Play

- Handout

Cont...

- Without graphing, determine whether each function represents exponential growth or exponential decay.
  - \( y = 100(0.12)^x \)
  - \( y = 0.2(5)^x \)
  - \( y = 16(\frac{1}{2})^x \)
Example

• Graphing Exponential Decay
  – Graph \( y = 24\left(\frac{1}{2}\right)^x \). Identify the horizontal asymptote.

Example

• Depreciation – The exponential decay graph shows the expected depreciation for a car over four years. Estimate the value of the car after six years.

<table>
<thead>
<tr>
<th>Years since purchase</th>
<th>Value ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>20000</td>
</tr>
<tr>
<td></td>
<td>17000</td>
</tr>
<tr>
<td></td>
<td>14450</td>
</tr>
<tr>
<td></td>
<td>12282.5</td>
</tr>
<tr>
<td></td>
<td>10440.125</td>
</tr>
<tr>
<td></td>
<td>5000</td>
</tr>
<tr>
<td></td>
<td>10000</td>
</tr>
<tr>
<td></td>
<td>15000</td>
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<tr>
<td></td>
<td>20000</td>
</tr>
<tr>
<td></td>
<td>25000</td>
</tr>
</tbody>
</table>

Example

• Graphing \( y = ab^x \) When \( a < 0 \)
  – Graph \( y = \frac{1}{2} \cdot 2^x \) and \( y = -\frac{1}{2} \cdot 2^x \). Label the asymptote of each graph.

Properties of Exponential Functions

Analyze, describe and sketch graphs of exponential functions by examining intercepts, zeros, domain and range, and asymptotic and end behavior.
Cont...

• Graph each function.
  - \( y = -4(2)^x \)
  - \( y = -3^x \)

Example

• Translating \( y = ab^x \)
  - Graph \( y = 8(\frac{1}{2})^x \) and \( y = 8(\frac{1}{2})^x + 2 + 3 \).

Cont...

• Graph each function as a translation of \( y = 9(3)^x \).
  - \( y = 9(3)^x + 1 \)
  - \( y = 9(3)^x - 4 \)
  - \( y = 9(3)^x - 3 - 1 \)

Example

• Medicine – The half-life of a radio active substance is the time it takes for half of the material to decay. A hospital prepares a 100-mg supply of technetium-99m, which has a half-life of 6 hours. Make a table showing the amount of technetium-99m that remains at the end of each 6-hour interval for 36 hours. Then write an exponential function to find the amount of technetium-99m that remains after 75 hours.

Cont...

• Arsenic-74 is used to locate brain tumors. It has a half-life of 17.5 days. Write an exponential decay function for a 90-mg sample. Use the function to find the amount remaining after 6 days.

Example

• Evaluating \( e^x \)
  - Graph \( y = e^x \). Evaluate \( e^2 \) to four decimal places.
Cont...

- Use the graph $y = e^x$ to evaluate each expression to four decimal places.
  - $e^4$
  - $e^{-3}$
  - $e^{1/2}$

Continuously Compounded Interest Formula

<table>
<thead>
<tr>
<th>Amount in account</th>
<th>Annual rate of interest</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$P e^{rt}$</td>
</tr>
<tr>
<td></td>
<td>$r$</td>
</tr>
<tr>
<td></td>
<td>$t$</td>
</tr>
<tr>
<td></td>
<td>Principal</td>
</tr>
</tbody>
</table>

Example

- Investments – Suppose you invest $1050 at an annual interest rate of 5.5% compounded continuously. How much will you have in the account after five years?

Cont...

- Suppose you invest $1300 at an annual interest rate 4.3% compounded continuously. Find the amount you will have in the account after three years.

Logarithmic Functions as Inverses

Know that the inverse of an exponential function is a logarithm. Use laws of exponents to derive laws of logarithms. Use the inverse relationship between exponential functions and logarithms and the laws of logarithms to solve problems.
Example

- **Seismology** – In 1995, an earthquake in Mexico registered 8.0 on the Richter scale. In 2001, an earthquake of magnitude 6.8 shook Washington state. Compare the amounts of energy released in the two earthquakes.

Example

- Writing in Logarithmic Form
  - Write $25 = 5^2$ in logarithmic form.

Example

- Evaluating Logarithms
  - Evaluating $\log_9 16$.

Cont...

- In 1997, an earthquake in Alabama registered 4.9 on the Richter scale. In 1999, one in California registered 7.0. Compare the energy released in the two quakes.

Cont...

- Write each equation in logarithmic form.
  - $729 = 3^6$
  - $(\frac{1}{2})^3 = \frac{1}{8}$
  - $10^0 = 1$

Cont...

- Evaluate each logarithm.
  - $\log_{64} 1/32$
  - $\log_9 27$
  - $\log_{10} 100$
Example

- Chemistry – The pH of lemon juice is 2.3, while the pH of milk is 6.6. Find the concentration of hydrogen ions in each substance. Which substance is more acidic?

Cont...

- Find the concentration of hydrogen ions in seawater of pH 8.5

Logarithmic Functions as Inverses

- A logarithmic function is the inverse of an exponential function.

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>$y = \log_b x$</th>
<th>$y = \log_b (x - h) + k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asymptote</td>
<td>$x = 0$</td>
<td>$x - h = 0$, or $x = h$</td>
</tr>
<tr>
<td>Domain</td>
<td>$x &gt; 0$</td>
<td>$x &gt; h$</td>
</tr>
<tr>
<td>Range</td>
<td>All real numbers</td>
<td>All real numbers</td>
</tr>
</tbody>
</table>

Example

- Translating $y = \log_b x$
  - Graph $y = \log_6 (x - 2) + 3$

Cont...

- Graph $y = \log_3 (x + 3)$. 

Properties of Logarithms
Investigation – Properties of Logarithms

• Handout

Properties of Logarithms

• For any positive numbers, $M$, $N$, and $b$, $b \neq 1$
  
  $\log_b MN = \log_b M + \log_b N$ -- Product Property
  
  $\log_b M/N = \log_b M - \log_b N$ -- Quotient Property
  
  $\log_b M^x = x \log_b M$ -- Power Property

Example

• Identifying the Properties of Logarithms
  
  State the properties used to rewrite each expression.
  
  $\log_2 8 - \log_2 4 = \log_2 2$
  
  $\log_b x^y = 3 \log_b x + \log_b y$

Cont...

• State the property or properties used to rewrite each expression.
  
  $\log_5 2 + \log_5 6 = \log_5 12$
  
  $3 \log_b 4 - 3 \log_b 2 = \log_b 8$

Example

• Simplifying Logarithms
  
  Write each logarithmic expression as a single logarithm.
  
  $\log_3 20 - \log_3 4$
  
  $3 \log_2 x + \log_2 y$

Cont...

• Write $3 \log 2 + \log 4 - \log 16$ as a single logarithm.

• Can you write $3 \log_2 9 - \log_6 9$ as a single logarithm? Explain.
Example

- Expanding Logarithms
  - Expand each logarithm.
    - \( \log_5 (x/y) \)
    - \( \log_3 a^4 \)

Cont...

- Expand each logarithm.
  - \( \log_2 7b \)
  - \( \log (y/3)^2 \)
  - \( \log_7 a^3b^4 \)

Example

- Noise Control – A shipping company has started flying cargo planes out of the city airport. Residents in a nearby neighborhood have complained that the cargo planes are too loud. Suppose the shipping company hires you to design a way to reduce the intensity of the sound by half. By how many decibels would the loudness of the sound be decreased?

Cont...

- Suppose the shipping company wants you to reduce the sound intensity to 25% of the original intensity. By how many decibels would the loudness be reduced?

Definitions

- An equation of the form \( b^x = a \), where the exponent includes a variable, is an exponential equation.
- Change of Base Formula
  - For any positive numbers, \( M, b, \) and \( c \), with \( b \neq 1 \) and \( c \neq 1 \),
    - \( \log_b M = \frac{\log_c M}{\log_c b} \)
- An equation that includes a logarithmic expression is called a logarithmic equation.
Example
• Solving an Exponential Equation
  – Solve $7^{3x} = 20$

Cont…
• Solve each equation. Check your answers.
  • $3^x = 4$
  • $6^{2x} = 21$
  • $3^{x+4} = 101$

Example
• Using the Change of Base Formula
  – Use the Change of Base Formula to evaluate $\log_3 15$. Then convert $\log_3 15$ to a logarithm in base 2.

Cont…
• Evaluate $\log_5 400$ and convert it to a logarithm in base 8.

Example
• Solving an Exponential Equation by Changing Bases
  – Solve $2^{3x} = 172$

Cont…
• Use the Change of Base Formula to solve $7^{5x} = 3000$. Check your answers.
Example

• Solving an Exponential Equation by Graphing
  – Solve $6^{2x} = 1500$

Cont…

• Solve $11^{6x} = 786$

Example

• Solving a Logarithmic Equation
  – Solve $\log (3x + 1) = 5$

Cont…

• Solve $\log (7 - 2x) = -1$. Check your answer.

Example

• Using Logarithmic Properties to Solve an Equation
  – Solve $2 \log x - \log 3 = 2$

Cont…

• Solve $\log 6 - \log 3x = -2$
THE END

Algebra II